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# CONTINUITY OF LOCALLY BOUNDED HOMOMORPHISMS OF CONNECTED LIE GROUPS INTO LIE GROUPS WITHOUT NONTRIVIAL COMPACT CONNECTED SUBGROUPS

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Abstract. We prove the automatic continuity of locally bounded homomorphisms of connected Lie groups into connected Lie groups without nontrivial compact connected subgroups.

## § 1. Introduction

In this note, we establish the continuity of every locally bounded homomorphism of a connected Lie group $G$ into a Lie group without nontrivial compact connected subgroups. In particular, the assertion holds for locally bounded homomorphisms into connected simply connected solvable Lie groups and for the homomorphisms into the universal covering group of $\mathrm{SL}(2,\mathbb{R})$.

## § 2. Preliminaries

Let us recall some information needed below.

A (not necessarily continuous) homomorphism $\pi$ of a topological group $G$ into a topological group $H$ is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element $e_G$ in $G$ whose image $\pi(U)$ has compact closure in $H$. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a

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neighborhood $U_e$ whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of $e_V$ having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism $\pi$ of a topological group $G$ into a topological group $H$, see [1] and [2]. Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of $e_G$ in $G$. For every (not necessarily continuous) locally relatively compact homomorphism $\pi$ of $G$ into $H$, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of $\pi$. Here and below, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of $H$). (See Definition 1.1.1 of [1].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group $H$ and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of $H$. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism $\pi$ is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. (See Theorem 1.1.2 of [1].) If $G$ is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of $H$. (See Lemma 1.1.6 of [1].)

## § 3. Main Result

**Theorem.** *Every locally bounded homomorphism of a connected Lie group $G$ into a Lie group without nontrivial compact connected subgroups is continuous.*

*Proof.* Let $\pi$ be a locally bounded homomorphism of a connected Lie group $G$ into a Lie group $H$ without nontrivial compact connected subgroups.

According to Lemma 1.1.6 of [1], the discontinuity group $\mathrm{DG}(\pi)$ is a compact connected subgroup of $H$.

Since, by the condition, there are no nontrivial compact connected subgroups of $H$, it follows that $\mathrm{DG}(\pi)$ is the identity subgroup of $H$.

By Theorem 1.1.2 of [1], this implies that the homomorphism $\pi$ is continuous, as was to be proved.

## § 4. Discussion

As a rule, a locally bounded homomorphism between Lie groups is continious only under additional conditions (for example, see [3–6].

Well-known examples of discontinuous finite-dimensional locally bounded linear representations of $\mathrm{sl}(2,\mathbb{R})$ can be regardsd as locally bounded linear representations of the universal covering group of $\mathrm{sl}(2,\mathbb{R})$. This group has no compact connected subgroups. Thus, a condition imposed in the theorem on the target group cannot be transferred to the source group: a Lie group with this property can have discontinuous locally bounded finite-dimensional representations.

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